Twisted topological structures related to M-branes

Hisham Sati [1]

Department of Mathematics
University of Maryland
College Park, MD 20742

Abstract

Studying the M-branes leads us naturally to new structures that we call Membrane-, Membrane^c-, String$^{K(\mathbb{Z},3)}$- and Fivebrane$^{K(\mathbb{Z},4)}$-structures, which we show can also have twisted counterparts. We study some of their basic properties, highlight analogies with structures associated with lower levels of the Whitehead tower of the orthogonal group, and demonstrate the relations to M-branes.

[1] e-mail: `hsati@math.umd.edu`

Contents

1 Introduction

This is a continuation of our study of geometric and topological structures related to M-branes in M-theory, and is the third paper in a series. In the first one [32] we outlined several ideas to be developed in later papers. In the second [34] we considered twisted String structures [44] and introduced twisted String^c structures associated with the M2-brane and the M5-brane. This builds on [37] where twisted String structures are considered in relation to the flux quantization condition in M-theory [46] and the Green-Schwarz anomaly cancellation in heterotic string theory. In addition, the paper [37] considered the dual picture, where the anomalies associated with the M5-brane and the dual of the Green-Schwarz anomaly cancellation lead to twisted Fivebrane structures, a twisted version of the Fivebrane structure introduced in [35] and studied in [36].

The obstructions for the String and Fivebrane cases are essentially the first and the second Pontrjagin classes, respectively. In this paper we consider, in addition, obstructions coming from the Stiefel-Whitney classes. The seventh integral Stiefel-Whitney class W_7 of spacetime appears as a condition for the partition function in dimensionally reduced M-theory to type IIA to be well-defined upon summing over torsion [7]. This was taken in [16] to provide an elliptic cohomology refinement of the K-theoretic partition function. In addition, such a construction led naturally to a condition on the mod 2 Steifel-Whitney class w_4. This has a natural interpretation in relation to the flux quantization of the C-field [32]. In this paper we provide an interpretation of these structures in terms of bundle constructions and also consider twists for such structures. The motivation comes from M-branes in M-theory, and so we find such structures in relation to worldvolumes, normal bundles of embedding to spacetime, and in target spacetime itself.

We define the following structures

1. *(Twisted) Membrane structures.* These are structures whose obstructions are degree four analogs of those of twisted Spin structures. Instead of having $w_2 + b = 0 \in H^2\mathbb{Z}_2$, we have the condition $w_4 + \alpha = 0 \in H^4\mathbb{Z}_2$, where α is a degree four $\mathbb{Z}_2$ class. This shows up in the quantization condition for the C-field and hence in relation to the partition function of the M2-brane, and also in the normal bundle of the M5-brane. We study this in section 3.

2. *(Twisted) Membrane^c structures.* A twisted membrane^c structure is a degree five analog of a twisted Spin^c structure, where instead of having the Freed-Witten condition $W_3 + H_3 = 0 \in H^3\mathbb{Z}$ [12], inter-

preted as the obstruction to having a twisted Spin^c structure [44] [8], we have $W_5 + H_5 = 0 \in H^5\mathbb{Z}$. Such a structure is also related to the C-field. We consider this in section 4.

3. *(Twisted) String$^{K(\mathbb{Z},3)}$ structures.* These are analogs of Spin^c structures in the sense that obstructions to their existence are given by an odd degree Steenrod operation on a cohomology class. Recall that the Spin^c condition is obtained by applying the Bockstein β on the Stiefel-Whitney class w_2 giving W_3. Now W_7 is similarly obtained via a Steenrod square applied to a characteristic class, namely Sq^3 acting on $\lambda = \frac{1}{2}p_1$. So we call such a structure a String$^{K(\mathbb{Z},3)}$ structure. We also consider the twist of such a structure and relate it to physical situations discussed in [30] and [32]. This is the subject of section 5.

4. *(Twisted) Fivebrane$^{K(\mathbb{Z},4)}$ structures.* String^c structure introduced in [5] are String structures corresponding to Spin^c bundles. Such structures are shown in [34] to be related to twisted String structures, and a twisted version was provided. We might call this a twisted String$^{K(\mathbb{Z},2)}$ in our present formalism. In degree eight, we find that we replace $K(\mathbb{Z},2)$ by $K(\mathbb{Z},4)$, the first Chern class of the line bundle corresponding to the Spin^c structure with the characteristic class a of an E_8 bundle, [2] and the String condition with the Fivebrane condition. Furthermore, this can be twisted by a degree eight cocycle. The result is what we call a twisted Fivebrane$^{K(\mathbb{Z},4)}$ structure, which we show is related to the dual C-field in M-theory, and hence to the M5-brane. This will be discussed in section 6.

The above structures admit restrictions to the boundary, which occur and are relevant in all cases, but we illustrate only for the case of twisted membrane structures, with the others deduced directly from that case. This is relevant for M-theory in the presence of a boundary, and for the M2-brane and the M5-brane, both in the presence of boundaries, and in treating them essentially as boundaries.

As the conditions appearing in quantization conditions and anomaly cancellation conditions are given only up to further denominators by such structures, we characterize the 'mismatch' from the point of view of cohomology operations in section 7. This should be viewed as complentary to the interpretations in [37] in terms of bundles encoding the extra congruences. In doing so we naturally highlight the importance of torsion. We find that the one-loop term in M-theory [9] [42] can be succinctly described using what we call *String characteristic classes*. Some of the relations between the various structures introduced in this paper are indicated in the sections where such structures are defined. Further connections are made in section 8.

2 Review of setting and basic notions

The goal of this section is to provide some motivation, setting, as well as tools and definitions which will be needed in the following sections.

Twisted Spin structure and type I fields. In the presence of a B-field B, type I D-branes can be wrapped on a submanifold D of spacetime only if [47]

$$w_2(D) + [B|_D] = 0 \in H^2(M;\mathbb{Z}_2) \ . \tag{2.1}$$

A geometric interpretation of such 2-torsion B-fields (also called t'Hooft classes) as the holonomy of connections for real bundle gerbes is given in [23], where both finite dimensional and infinite dimensional geometric realizations are given. In [44] the condition (2.1) is interpreted as twisted Spin structure.

Twisted Spin^c structures and the Ramond-Ramond fields. In the presence of a Neveu-Schwarz (NS) B-field, type II D-branes can be wrapped on a submanifold D of spacetime only if [12]

$$W_3(D) + [H|_D] = 0 \in H^3(M;\mathbb{Z}) \ . \tag{2.2}$$

[2]Note that $E_8 \sim K(\mathbb{Z},4)$ in our range of dimensions.

A geometric interpretation in terms of bundle gerbes is given in [4]. This condition (2.2) is not sufficient and further obstructions arise at primes higher than 2 [10].

Generalizations of the above structures to higher levels in the Whitehead tower of the homotopy groups of the orthogonal group are also relevant. In [44] the notion of twisted String structure was defined. This was refined in [37] to the differential case, where also the physical applications to the C-field and to the Green-Schwarz anomaly formula are discussed. Furthermore, the notion of twisted Fivebrane structure was introduced in [37], with the dual C-field and the dual Green-Schwarz anomaly formula viewed as (essentially) obstructions to such a structure.

Twisted String structure and the C-field. The C-field, via its field strength G_4, in M-theory on a Spin eleven-manifold Y satisfies the quantization condition [46]

$$[G_4] - \frac{\lambda}{2} = a \in H^4(Y; \mathbb{Z}) \,, \tag{2.3}$$

where $\lambda = \frac{1}{2}p_1$ is half the first Pontrjagin class of Y (see (2.11)) and a is the degree four characteristic class of an E_8 bundle over Y. We will sometimes omit the notation $[x]$ for a cohomology class, and use just x instead. [3] The corresponding geometric structure is a twisted String structure [37], in the sense of [44].

Twisted Fivebrane structures and the dual C-field. The dual C-field, via its field strength G_8, in M-theory on Y satisfies the quantization condition [6]

$$\Theta := [G_8] = \frac{1}{48}p_2 + \beta \,, \tag{2.4}$$

where β is a degree eight class. This is interpreted as a twisted Fivebrane structure, defined in [37]. The untwisted case, in which β is not present corresponding to $\frac{p_2}{6} = 0$ was introduced in [35] [36].

The factors in the denominators of the Pontrjagin classes in (2.3) and (2.4) are interpreted in terms of twisted String and Fivebrane structure, respectively, which are modified in the appropriate sense [37]. This will be studied further in this paper, in section 7, from the point of view of cohomology operations. We will mostly deal with torsion in cohomology.

Description of the higher Stiefel-Whitney classes. Since there are flat manifolds M with nonzero Stiefel-Whitney classes $w_k(TM)$ [1], one cannot hope for an analog of Chern-Weil theory for Stiefel-Whitney classes. However, one can hope for a Čech description. The cases $k = 1, 2$ are well-known (see [21]). An explicit Čech cocycle representing the kth Stiefel-Whitney class of an n-dimensional vector bundle over a manifold M is given in [26] using topological $\mathbb{Z}_2$ Deligne cohomology, a refinement of usual sheaf cohomology. The formula involves only the transition functions of the bundle relative to some trivializing open cover $\{U_i\}$ of M. For each point x in $U_{i_0} \cap \cdots \cap U_{i_k}$, there is a $(k-1)$-cycle on $\mathrm{SO}(n)/\mathrm{SO}(k-1)$ which depends continuously on x. The vertices of this cycle are defined by the value at x of the transition functions of the bundle. A class in $H^k(M; \mathbb{Z}_2)$ is defined by sending x to 1 if this cycle is homologically nontrivial, and to 0 otherwise. This class coincides with $w_k(FE)$, where FE is the frame bundle of E. Another description views the Stiefel-Whitney classes as representing obstructions to orientations with respect to generalized cohomology theories. The cases most relevant to this article, namely w_4 and W_7, are worked out in [16].

Now we recall some tools (see e.g. [28]).

Reduction in cohomology. Let $\rho_i : \mathbb{Z} \to \mathbb{Z}_j$ denote mod j reduction, that is $\rho_j(1) = 1$ mod j, for $j = 2, 3, \cdots$. We are mainly interested in the case $j = 2$, corresponding to mod 2 reduction, and the case $j = 4$, corresponding to mod 4 reduction (see section 7). We use the same notation for the cohomology

[3] This should not cause confusion since we will not deal with differential form representatives in this paper.

homomorphism induced by ρ_i. Let δ_* be the Bockstein coboundary associated with the exact sequence $0 \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \xrightarrow{\rho_k} \mathbb{Z}_k \to 0$ (of coefficients) and set $\beta_k = \rho_k \delta_*$.

Mod 2 reduction. The coboundary β_2 is a derivation, that is for any two mod 2 cohomology classes u and v, the relation $\beta_2(uv) = \beta_2(u)v + u\beta_2(v)$ holds. The action of β_2 on the Stiefel-Whitney classes is

$$\beta_2(w_{2i}) = w_1 w_{2i} + w_{2i+1} \ , \qquad \beta_2(w_{2i+1}) = w_1 w_{2i+1} \ . \tag{2.5}$$

We are dealing with oriented bundles, so the second equation will not be of relevance to us, while the first equation reduces to $\beta_2(w_{2i}) = w_{2i+1}$. An example of this, for $i = 3$, is equation (5.1) since $Sq^2 w_4 = w_6$.

Action of Steenrod squares on Stiefel-Whitney classes. There are two flavors of the Steenrod square cohomology operation. The first one is a mod 2 to integral cohomology operation

$$Sq^i_{\mathbb{Z}} : H^k(X;\mathbb{Z}_2) \to H^{k+i}(X;\mathbb{Z}) \ , \tag{2.6}$$

and the second one is a mod 2 to mod 2 cohomology operation

$$Sq^i : H^k(X;\mathbb{Z}_2) \to H^{k+i}(X;\mathbb{Z}_2) \ . \tag{2.7}$$

The two are related by $Sq^i_{\mathbb{Z}} = \beta Sq^{i-1}$. The action of the Steenrod square on the Stiefel-Whitney classes of a bundle E is given by the Wu formula

$$Sq^i w_{i+1}(E) = \sum_{j=0}^{i} w_j(E) w_{2i+1-j}(E) \ . \tag{2.8}$$

For $i = 1, 2$, and 3, respectively, this gives

$$\begin{aligned} Sq^1 w_2 &= w_1 w_2 + w_3 \ , \\ Sq^2 w_3 &= w_5 + w_1 w_4 + w_2 w_3 \ , \\ Sq^3 w_4 &= w_7 + w_1 w_6 + w_2 w_5 + w_3 w_4 \ . \end{aligned} \tag{2.9}$$

Note that when E is oriented and Spin, we have $w_3(E) = 0$, $w_5(E) = 0$ and so $Sq^3 w_4(E) = w_7(E)$. It is generic that the interesting even degree classes are the mod 2 ones, w_{2k}, while interesting odd degree classes are images of these in integral cohomology, i.e. $Sq^i_{\mathbb{Z}} w_{2k} = W_{2k+i}$ (for appropriate i).

Spin characteristic classes When a bundle E is Spin the corresponding characteristic classes will admit special values. The second Stiefel-Whitney class $w_2(E)$ will vanish, and the first Pontrjagin class $p_1(E)$ will be divisible by two because of the relation $p_1(E) = w_2(E)^2$ mod 2. Then, naturally, the bundle would be described via characteristic classes related to the Spin group rather than to the orthogonal group. The integral cohomology ring of the classifying space of the Spin group is [40]

$$H^*(B\mathrm{Spin};\mathbb{Z}) \cong \mathbb{Z}[Q_1, Q_2, \cdots] \oplus T \ , \tag{2.10}$$

where the generators $Q_i \in H^{4i}(B\mathrm{Spin};\mathbb{Z})$ are the Spin characteristic classes, defined via the corresponding Pontrjagin classes by

$$p_1 = 2Q_1 \ , \qquad p_2 = Q_1^2 + 2Q_2 \ . \tag{2.11}$$

The summand T in (2.10) is 2-torsion, $2T = 0$. For example, in dimension seven this is generated by the seventh Stiefel-Whintey class $W_7 = \delta w_6$, with $\delta : H^*(X;\mathbb{Z}_2) \to H^*(X;\mathbb{Z})$ the Bockstein on cohomology. Mod 2 characteristic classes of Spin bundles are obtained by pullback from the universal classes generating the cohomology ring of BSpin [29]

$$H^*(B\mathrm{Spin};\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, w_{10}, \cdots] \ . \tag{2.12}$$

Obviously, all classes w_i up to $i = 3$ are absent. The degree four Spin characteristic class $Q_1 = \lambda := \frac{1}{2}p_1$ admits $w_4(E)$ as a mod 2 reduction

$$\rho_2(Q_1(E)) = w_4(E) \ . \tag{2.13}$$

Generally, the classes Q_i admit the Stiefel-Whitney classes in the same dimension as mod 2 reductions $\rho_2(Q_i) = w_{4i}$. Here ρ_2 is the induced homomorphism on cohomology arising from the corresponding reduction in coefficients $\rho_2 : \mathbb{Z} \to \mathbb{Z}_2$. We will see more of this in section 7.

3 (Twisted) Membrane Structures

We now define the first new structure considered in this paper.

Membrane structure. The Stiefel-Whitney class w_4 is the mod 2 reduction of $\lambda = \frac{1}{2}p_1$. This implies that λ is even if and only if $w_4 = 0$. We use this as the obstruction to having a *Membrane structure*; that is a Membrane structure on a bundle E can be defined when $w_4(E) = 0 \in H^4(E; \mathbb{Z}_2)$. This is closely related to a String structure in the following sense. If $\lambda(E) = 0$ then $w_4(E) = 0$; this means that String bundles are automatically Membrane bundles. However, certainly there are bundles E for which $w_4(E) = 0$ but $\lambda(E) \neq 0$; for instance, $\lambda(E)$ instead is an even class.

Additive structure in the untwisted case. If at least one of two real bundles E, F, is oriented and Spin, then for the Whitney sum we have

$$\begin{aligned} w_4(E \oplus F) &= w_4(E) + w_3(E)w_1(F) + w_2(E)w_2(F) + w_1(E)w_3(F) + w_4(F) \\ &= w_4(E) + w_4(F) \ . \end{aligned} \tag{3.1}$$

Here we use the fact that the first nontrivial Stiefel-Whitney class should be in even degree (cf. (2.8)). Therefore, in particular, the product of two oriented Spin Membrane manifolds is again an oriented Spin Membrane manifold.

Application: Membrane structures related to the M5-brane. Consider the M5-brane with worldvolume $\mathcal{W}^6$ embedded in eleven-dimensional spacetime Y, with normal bundle $\mathcal{N}$. Let $S(\mathcal{N})$ be the unit sphere bundle of $\mathcal{N}$ of dimension ten and $\pi : S(\mathcal{N}) \to \mathcal{W}^6$ the projection. Let a be a degree four class on $S(\mathcal{N})$. Then [48]

$$\pi_*(a \cup a) \cong w_4(\mathcal{N}) \mod 2 \ . \tag{3.2}$$

It is desirable that the left hand side of (3.2) be even so that the partition function is well-defined [48]. We see that this condition is satisfied when $w_4(\mathcal{N}) = 0$, i.e. if the normal bundle to the M5-brane admits a Membrane structure. One can also get such a structure on the worldvolume, under some conditions. For example, if $\mathcal{W}^6$ is a 2-connected six-manifold then we have $H^4(\mathcal{W}^6; \mathbb{Z}_2) = 0$. In particular, $w_4(\mathcal{W}^6) = 0$, so that indeed the M5-brane worldvolume admits a Membrane structure.

Next, we consider the twist for the membrane structure, in analogy to other structures [44] [37].

The definition. Let (X, α) be a compact topological space with a degree four cocycle $\alpha : X \to K(\mathbb{Z}_2, 4)$. An *$\alpha$-twisted Membrane-manifold* over X is a quadruple (M, ν, ι, η), where
(1) M is a smooth compact oriented manifold together with a fixed classifying map of its stable normal bundle $\nu : M \to B\mathrm{SO}$;
(2) $\iota : M \to X$ is a continuous map;
(3) η is an α-twisted Membrane-structure on M, that is, a homotopy commutative diagram

$$
\begin{tikzcd}
M \arrow[r, "\nu"] \arrow[d, "\iota"'] & BSO \arrow[d, "w_4"] \arrow[dl, Rightarrow, "\eta"] \\
X \arrow[r, "\alpha"'] & K(\mathbb{Z}_2, 4)
\end{tikzcd}
\quad , \tag{3.3}
$$

where w_4 is the classifying map of principal $K(\mathbb{Z}_2, 3)$ bundles associated to the fourth Stiefel-Whitney class, and η is a homotopy between $w_4 \circ \nu$ and $\alpha \circ \iota$.

Remarks. Given a smooth compact oriented n-manifold M and a topological space X with a twisting $\alpha : X \to K(\mathbb{Z}_2, 4)$, then

1. M admits an α-twisted Membrane-structure if and only if there exists a continuous map $\iota : M \to X$ such that
$$w_4(M) + \iota^*([\alpha]) = 0 \in H^4(M; \mathbb{Z}_2) \; . \tag{3.4}$$
2. If (3.4) is satisfied, then the set of equivalence classes of α-twisted Membrane-structures on M are in one-to-one correspondence with elements in $H^3(M; \mathbb{Z}_2)$.
3. If the twisting $\alpha : X \to K(\mathbb{Z}_2, 4)$ is homotopic to the trivial map then an α-twisted Membrane structure on M is equivalent to a Membrane structure on M.
4. Two α-twisted Membrane structures η and η' on M are equivalent if there is a homotopy between η and η'.

Additive structure in the twisted case. Let α be the same as in diagram (3.3). An example of such an α occurs by considering fractional G_4 flux. The same result as in the untwisted case holds. That is, for a product manifold X whose tangent bundle splits as $TX = E \oplus F$ and with a twisting $\alpha : X \to K(\mathbb{Z}_2, 4)$, we have

$$
\begin{aligned}
(w_4 + \alpha)(E \oplus F) &= w_4(E \oplus F) + \alpha(E \oplus F) \\
&= w_4(E) + w_4(F) + \alpha(E) + \alpha(F) \\
&= (w_4 + \alpha)(E) + (w_4 + \alpha)(F) \; .
\end{aligned} \tag{3.5}
$$

Therefore, the product of two oriented Spin twisted Membrane manifolds is also an oriented Spin twisted Membrane manifold.

Examples. A manifold M is a boundary if and only if all its Stiefel-Whitney numbers vanish (see [38]). Since every flat manifold is a boundary [13], all Stiefel-Whitney numbers of a flat manifold vanish. However, the same is not true for the corresponding classes. Indeed, there are examples of flat manifolds with non-vanishing Stiefel-Whitney classes. Consider toral extensions, i.e. torus bundles over flat manifolds, which can certainly arise in realistic physical models (cf. [33]), have all the even-dimensional Stiefel-Whitney classes nonzero up to the middle dimension. So in the case of string theory or M-theory, we get nonvanishing of w_2 and w_4. Such manifolds can be desribed as follows. Let $Q \subset \mathbb{R}^n \ltimes O(n)$ be a Bieberbach group so that it acts on $\mathbb{R}^n$ freely and properly discontinuously. Suppose Q also acts on the torus T^k as isometries so that it acts on the product $T^k \times \mathbb{R}^n$ diagonally as isometries. The quotient $M = (T^k \times \mathbb{R}^n)/Q$ is called [43] a *flat toral extension* of the compact flat Riemannian manifold $N = \mathbb{R}^n/Q$. Then M is a torus bundle over N.

There are other examples which are not toral extensions [15]. There is a class of $(2n+1)$-dimensional *compact* flat manifolds whose Stiefel-Whitney classes w_{2j} are non-zero for $0 \leq 2j \leq n$, none of which is a flat toral extension of another flat manifold. This manifold M has holonomy group $(\mathbb{Z}_2)^{n+1}$ and has a vanishing first Betti number $b_1(M) = 0$. Taking $n = 5$ we get eleven-manifolds with nonzero w_2 and w_4 with the above properties.

Interpretation of the w_4 condition. The w_4 condition arises as an orientation condition with respect to $EO(2)$-theory, needed to construct an anomaly-free partition function in type IIA string theory [16]. On the other hand, Stiefel-Whitney class of dimension close to the dimension of the space take on interesting roles. For example, when the manifold M is 4-dimensional, w_4 admits a special interpretation, related to global causality. In [11] it is shown that if spacetime (M, g) is stably causal, then $w_4(M) = 0$, and there exists a 5-manifold V such that $M = \partial V$.

Application: Twisted Membrane structure associated to the M2-brane. If $\lambda(M)$ is not even then $w_4(M) \neq 0 \in H^4(M; \mathbb{Z}_2)$. Then there exists a class α (related to G_4, cf. expression (2.3)) such that $w_4 + \alpha = 0 \in H^4(M; \mathbb{Z}_2)$. This is the case for the quantization of the C-field when λ is not divisible by two. Therefore, a twisted Membrane structure arises naturally from considering M-theory on manifolds with an odd first Spin class.

Application: Twisted Membrane structure associated to the M5-brane. Consider the situation of M-theory on $\mathbb{R}^5/\mathbb{Z}_2$, giving rise to an M5-brane [46]. A four-cycle surrounding the origin in this space can be taken to be the real projective space $\mathbb{R}P^4 = S^4/\mathbb{Z}_2$. The mod 2 cohomology ring of $\mathbb{R}P^4$ is a polynomial ring in a degree one generator x with relation $x^5 = 0$. This gives that x^4 is the mod 2 fundamental class of $\mathbb{R}P^4$, so that $\int_{\mathbb{R}P^4} w_4 = 1$ mod 2. Thus, there is a half-integral flux of G_4. A similar situation arises when considering the $\mathbb{R}^5/\mathbb{Z}_2$ orbifold with a $\mathbb{Z}_2$ fixed plane along a Riemann surface Σ [14]. This gives the integral over the relevant 4-cycle $\int_S w_4 = \chi(L)$ mod 2, where $\chi(L)$ is the Euler characteristic of a line bundle over Σ. We can see that when $\chi(L)$ is odd then $w_4 = 1$. In both cases, we get that there exists a class α such that $w_4 + \alpha = 0 \in H^4(M; \mathbb{Z}_2)$, that is we have a twisted Membrane structure.

Boundary case. Let (M, ν, ι, η) be an α-twisted Membrane manifold over X. Then there is a natural α-twisted Membrane structure on the boundary ∂M with outer normal orientation which is the restriction of the α-twisted Membrane structure on M

$$\begin{tikzcd} \partial M \arrow[r, "\nu|_{\partial M}"] \arrow[d, "\iota|_{\partial M}"'] & B\mathrm{Spin} \arrow[d, "w_4"] \arrow[dl, Rightarrow, "\eta_{\partial M}"] \\ X \arrow[r, "\alpha"'] & K(\mathbb{Z}_2, 4) \end{tikzcd} . \tag{3.6}$$

Examples. M-theory can be formulated on a manifold with boundary, and the quantization condition for the C-field extends to the boundary (cf. [6]). Hence, we can consider the structures we define in this paper restricted to that boundary. Similarly for the M2-brane and the M5-brane; they admit boundaries on the M5-brane and the M9-brane, respectively (see [39] [41]). In addition to the case of the twisted Membrane structure explicitly considered above, the other structures restrict to the boundary in a similar way, with the obvious changes to diagram (3.6).

Remarks on orientation. **1.** Let $\tau_X : X \to B\mathrm{Spin}$ be the classifying map of the stable tangent bundle of X. Then a $w_4 \circ \tau_X$-twisted Membrane structure on M is equivalent to an $EO(2)$-oriented map from M to X.
2. Let (M, ν, ι, η) be an α-twisted Membrane manifold over X. Any $EO(2)$-oriented map $f : M' \to M$ defines a canonical α-twisted Membrane structure on M'.

4 (Twisted) Membrane^c Structures

Membrane^c structures. We define a Membrane^c structure in an analogous way to a Spin^c structure, where instead of using W_3 we use $W_5 = \beta w_4$. Recall that a Membrane structure is defined when the

obstruction w_4 vanishes. Now if this obstruction is not zero but is a mod 2 reduction of an integral class (which would be Q_1 in the Spin case) then W_5 is zero. We take this as defining the obstruction to having a Membrane^c structure.

We now consider the twisted case.

Twisted Membrane^c structures. Let $f : X \to BO$ be the classifying map for the orthogonal bundle over X. An H_5-twisted Membrane^c structure on a space X is defined by the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "H_5"'] & BO \arrow[d, "W_5"] \arrow[dl, phantom, "\swarrow\eta"] \\ & K(\mathbb{Z},5) \end{tikzcd} \quad , \tag{4.1}$$

where the homotopy η is between the map $f \circ W_5$ ($f^*W_5 = W_5(X)$) and the five-cocycle corresponding to the class $H_5 \in H^5(X;\mathbb{Z})$. We can also include a brane as in diagram (3.3) and the extension is obvious.

Example/Application. We consider a Membrane^c structure with a trivial twist. The class $W_5(Y) \in H^5(Y;\mathbb{Z})$ is obtained from the Bockstein homomorphism applied to $w_4(Y)$. The degree five class is interpreted in [6] as the background magnetic charge induced by the topology of Y, and must vanish in order to be able to formulate any (electric) C-field. When Y is Spin, $W_5(Y) = 0$, since the class λ is an integral lift of $w_4(Y)$. Similarly, the obstruction to existence of a $Sp(2)$ bundle and to a global $\mathrm{Spin}(1,5)$ bundle on the M5-brane wolrdvolume $\mathcal{W}^6$ cancel if $W_5(\mathcal{W}^6) = 0$ (cf. [19]).

5 Twisted String$^{K(\mathbb{Z},3)}$ Structures

String$^{K(\mathbb{Z},3)}$ structures. These structures are defined using the seventh Stiefel-Whitney class. Let us start by mentioning a distinction. For the mod 2 Stiefel-Whitney classes we have

$$Sq^1 Sq^2 w_4 = Sq^3 w_4 = w_7 \in H^7\mathbb{Z}_2 \ , \tag{5.1}$$

while for the Spin characteristic class $\lambda = Q_1$ we have

$$Sq^3 \lambda = Sq^3 Q_1 = W_7 \in H^7\mathbb{Z} \ . \tag{5.2}$$

We define a String$^{K(\mathbb{Z},3)}$ structure on a manifold M by the condition $W_7(M) = 0$.

Application. The DMW anomaly [7] for dimensionally-reduced M-theory partition function to be well-defined is given exactly by $W_7 = 0$. This is discussed extensively in [16].

Examples. **1.** An example which is a String$^{K(\mathbb{Z},3)}$-manifold but not a String manifold is $X^{10} = S^2 \times S^2 \times \mathbb{C}P^3$. This has a nonzero λ (non-torsion), while there is no odd cohomology, so that $W_7 = 0$.
2. An example which is neither String nor String$^{K(\mathbb{Z},3)}$ is the eight-dimensional Spin homogeneous space of Lie groups $G_2/SO(4)$. This has nonzero Stiefel-Whitney classes w_4, w_6 and w_8 (see [3]). The class w_6 gives that $W_7(G_2/SO(4)) \neq 0$, so that the space is not String$^{K(\mathbb{Z},3)}$. On the other hand, the Pontrjagin classes are $p_1(G_2/SO(4)) = 2$ and $p_2(G_2/SO(4)) = 7$, so that the first Spin characteristic class is $Q_1(G_2/SO(4)) = 1$ (indeed $w_4(G_2/SO(4)) = 1$). This implies that $G_2/SO(4)$ is not String.

Additive structure in the untwisted case. We consider the class of a Whitney sum of two bundles E and F,

$$\begin{aligned} W_7(E\oplus F) &= \beta w_6(E\oplus F) \\ &= \beta\left[w_6(E)+w_5(E)w_1(F)+w_4(E)w_2(F)+w_3(E)w_3(F)+\right. \\ &\qquad \left. +\,w_2(E)w_4(E)+w_1(E)w_5(F)+w_6(F)\right]\ . \end{aligned} \tag{5.3}$$

1. If E and F are both oriented then $w_1(E)=0=w_1(F)$ and the two terms containing w_1 will vanish.
2. If E and F are both Spin then, in addition, $w_2(E)=0=w_2(F)$, so that all cross-terms in (5.3) vanish, leaving
$$W_7(E\oplus F)=W_7(E)+W_7(F)\ . \tag{5.4}$$
Here we used the the Wu formula (2.9).
3. The same conclusion holds if E and F are not Spin but only $EO(2)$-orientable, that is if $w_4(E)=0=w_4(F)$.

Example. Consider a product manifold $Z=X\times Y$. Then the tangent bundle of Z is the Whitney sum of the tangent bundles of X and Y, so that we can apply (5.4). If neither X nor Y have a String$^{K(\mathbb{Z},3)}$ structure, then this means that $W_7(X)=W_7(Y)=1$, and (5.4) gives that $W_7(Z)=0$, since W_7 is 2-torsion. This then says that out of two manifolds neither of which is String$^{K(\mathbb{Z},3)}$, we can build a third manifold in a straightforward way, namely their product, which is String$^{K(\mathbb{Z},3)}$.

Next we consider a twist for a String$^{K(\mathbb{Z},3)}$ structure.

The definition. Let (X,H_7) be a compact topological space with a degree seven cocycle $H_7: X\to K(\mathbb{Z},7)$. An *$H_7$-twisted String$^{K(\mathbb{Z},3)}$-manifold* over X is a quadruple (M,ν,ι,η), where
(1) M is a smooth compact Spin manifold together with a fixed classifying map of its stable normal bundle $\nu: M\to B\mathrm{Spin}$;
(2) $\iota: M\to X$ is a continuous map;
(3) η is a H_7-twisted String$^{K(\mathbb{Z},3)}$-structure on M, that is, a homotopy commutative diagram

$$\begin{tikzcd} M \arrow[r, "\nu"] \arrow[d, "\iota"'] & B\mathrm{Spin} \arrow[d, "W_7"] \arrow[dl, Rightarrow, "\eta"] \\ X \arrow[r, "H_7"'] & K(\mathbb{Z},7) \end{tikzcd}\ . \tag{5.5}$$

Additive structure in the twisted case. Now consider the twisted class $W_7^H:=W_7+H_7$. Consider the value on the Whitney sum of two bundles E and F. Similarly to the case of twisted Membrane structures (cf. equation (3.5)), for a product manifold X with a splitting of the tangent bundle $TX=E\oplus F$, this gives

$$W_7^H(E\oplus F)=W_7^H(E)+W_7^H(F) \tag{5.6}$$

if the conditions as in the twisted case above are satisfied.

Application. The H_7 twist appears at the level of de Rham cohomology in the dual formulation of heterotic string theory [30]. Since elliptic genera are needed to calculate anomalies in heterotic string theory (see [22]), and since dualities connect this theory to type II theories and M-theory where elliptic refinements occur [16][17] [18], this all strongly gives that elliptic cohomology is a natural tool to study the heterotic theory. A further indication is the requirement of a string structure [45] [20]. Then, the existence of the above twist should admit a lift to generalized cohomology [30], where the condition W_7+H_7 would appear. Note also that we have the twisted String$^{K(\mathbb{Z},3)}$ structure already from the twisted String condition (albeit in a trivial way), as we show in section 8.

6 Twisted Fivebrane$^{K(\mathbb{Z},4)}$ Structures

Fivebrane$^{K(\mathbb{Z},4)}$ structures. Recall that in [34] we interpreted a String^c structure as a twisted String structure with a product twist, i.e. that is coming from the cup product of two elements from $K(\mathbb{Z},2)$. So then a String^c structure might alternatively be called a String$^{K(\mathbb{Z},2)}$ structure. Note that the twist is still a degree four class, but taken to be a composite, i.e. built out of a product of two copies of (the same) degree two element.

Similarly in the Fivebrane case, we can define a Fivebrane$^{K(\mathbb{Z},4)}$ structure to correspond to a twisted Fivebrane structure where the degree eight twist is a composite of two degree four twists.

We need the following for the definition below (see [24] or [25]). For any based space X, the suspension with the $S^0 = \{0,1\}$ gives $S^0 \wedge X = X$. This shows that every spectrum is canonically a module over the sphere spectrum. Taking X to be the Eilenberg-MacLane spectrum $(H\mathbb{Z})_n = K(\mathbb{Z},n)$, gives that $S^0 \wedge K(\mathbb{Z},n) = K(\mathbb{Z},n)$. This gives rise to a morphism $s : S^0 \to K(\mathbb{Z},n)$ which, together with the identity map $\mathrm{id} : K(\mathbb{Z},n) \to K(\mathbb{Z},n)$, gives a product map $s \times \mathrm{id} : S^0 \times K(\mathbb{Z},n) \to K(\mathbb{Z},n) \times K(\mathbb{Z},n)$. This is turn induces a map $S^0 \wedge K(\mathbb{Z},n) \to K(\mathbb{Z},n) \wedge K(\mathbb{Z},n)$, which in turn gives a map $\wedge : K(\mathbb{Z},n) \to K(\mathbb{Z},n) \wedge K(\mathbb{Z},n)$.

The definition. A *Fivebrane*$^{K(\mathbb{Z},4)}$ *structure* on a space X with a String structure classifying map f with a degree eight cocycle α is characterized by homotopy between the String class $\frac{1}{6}p_2$ and the composite cocycle α. The cocycle α is a cup product of two degree four cocycles defined via the map $l_4 : K(\mathbb{Z},4) \times K(\mathbb{Z},4) \to K(\mathbb{Z},8)$, which classifies the cup product operation $H^4(X;\mathbb{Z}) \times H^4(X;\mathbb{Z}) \to H^8(X;\mathbb{Z})$. More precisely, we have the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a(X)"'] \arrow[ddr, "\alpha"] & B\mathrm{String}(n) \arrow[dd, "\frac{1}{6}p_2"] \arrow[d, phantom, "\Swarrow \eta_1"] \\
K(\mathbb{Z},4) \arrow[d, "\wedge"] \arrow[dr, phantom, "\Swarrow \eta_2"] & {} \\
K(\mathbb{Z},4) \wedge K(\mathbb{Z},4) \arrow[r, "\cup"] & K(\mathbb{Z},8)
\end{tikzcd} \quad . \tag{6.1}$$

The first homotopy η_1 gives the relation $\frac{1}{2}p_2 + \alpha = 0 \in H^8(X;\mathbb{Z})$ and the second homotopy η_2 gives $\alpha + \frac{1}{2}a^2 = 0 \in H^8(X;\mathbb{Z})$. Combined, the two homotopies then give

$$\frac{1}{6}p_2 + \frac{1}{2}a^2 = 0 \in H^8(X;\mathbb{Z}) \ . \tag{6.2}$$

This identifies a Fivebrane$^{K(\mathbb{Z},4)}$ structure as a special case of a twisted Fivebrane structure of [37].

Application: The dual C-field in M-theory. The equation of motion for the C-field in M-theory has an electric source which is an eight-form. In the case when the background manifold admits a Spin structure – which is the simplification assumed also in [37]– the dual class is (cf. (2.4))

$$\frac{1}{48}p_2(Y) + \frac{1}{2}a^2 \ . \tag{6.3}$$

Note the extra factor of 8 in the denominator of the coefficient of $\frac{1}{6}p_2$ in (6.3) in comparison to the obstruction to the Fivebrane$^{K(\mathbb{Z},4)}$ structure (6.2). The structure described by (6.3) should be an $\mathcal{F}_{\langle 8\rangle}^{K(\mathbb{Z},4)}$ structure. Here $\mathcal{F}_{\langle 8\rangle}$ is the object fibering over BString whose class is $\frac{1}{48}p_2$ instead of $\frac{1}{6}p_2$, as explained in [36][37]. An alternative point of view would be to perform reductions mod 8 via cohomology operations. We will do the analog for the String case next in section 7.

7 Congruences for String and Fivebrane classes

The quantization of the C-field and the Green-Schwarz anomaly cancellation involves a fractional first Pontrjagin class, and similarly the dual C-field and the dual Green-Schwarz cancellation involves a fractional second Pontrjagin class [36][37]. Furthermore, these classes do not precisely match String and Fivebrane structures, respectively, but only up to further divisions [36] [37]. Here we study such division (or fractions) from the point of view of cohomology operations.

7.1 String class mod 2

In this section we will consider the class $\frac{1}{2}\lambda = \frac{1}{2}Q_1 = \frac{1}{4}p_1$ appearing the flux quantization condition of the C-field (2.3). For this, we will study the mod 4 reduction of the first Pontrjagin class. We start with setting up some notation and providing some basic definitions.

Mod 4 reduction. The epimorphism $\rho_4 : \mathbb{Z} \to \mathbb{Z}_4$ induces a homomorphism of cohomology groups $H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}_4)$, $\alpha \longmapsto \rho_4(\alpha)$. Let us illustrate this for the Pontrjagin classes. A theorem of Wu says that for any $O(m)$ bundle E over X, the class $\rho_4(p_k(E)) \in H^{4k}(X;\mathbb{Z}_4)$ is determined by the Stiefel-Whitney classes $w_l(E) \in H^l(X;\mathbb{Z}_2)$. In particular, if the Stiefel-Whitney classes $w_1(E), \cdots, w_{k-1}(E)$ are zero then $\rho_4(p_k(E)) = i_{2*}w_{4k}(E)$, where i_2 is the homomorphism induced by the inclusion of coefficients $i_2 : \mathbb{Z}_2 \to \mathbb{Z}_4$.

Pontrjagin squares. Next, let the inclusion $i_j : \mathbb{Z}_2 \to \mathbb{Z}_{2^j}$, $i \in \mathbb{N}$, be defined by $i_j(1 \text{ mod } 2) = 2^{j-1} \text{ mod } 2^j$. We are interested in mod 2^m reduction of cohomology classes. The relevant cohomology operation is the Pontrjagin square. This is the cohomology operation (cf. [28])

$$\mathfrak{P} : H^{2k}(X;\mathbb{Z}_2) \to H^{4k}(X;\mathbb{Z}_4) \ , \tag{7.1}$$

satisfying the relation $\mathfrak{P}\rho_2(x) = \rho_4(x^2)$, for $x \in H^{2k}(X;\mathbb{Z})$.

We will explore the mod 4 reduction of the first Pontrjagin class in various situations. M-theory can be considered on Spin^c manifolds, and the flux quantization extends. We can also consider complex manifolds (as factors) as well as complex vector bundles. Note that the quantization condition extends also to the worldvolume of the M5-brane, which need not be Spin or even Spin^c (see [48] [34] for discussions on these matters).

Mod 4 reduction for characteristic classes of bundles. Let E be a bundle with some characteristic class(es).
1. Real bundles. First take E to be an n-dimensional real vector bundle over a space X. Then the mod 2 reduction of its Pontrjagin class is

$$\rho_2(p_1(E)) = w_2(E)^2 \ . \tag{7.2}$$

On the other hand, the mod 4 reduction satisfies $\mathfrak{P}(w_2) = \rho_4(p_1) + i_2[w_1 Sq^1 w_2 + w_4]$, where $i_2 : \mathbb{Z}_2 \to \mathbb{Z}_4$ is given by $i_2(1 \text{ mod } 2) = 2 \text{ mod } 4$. If the bundle E is oriented, then

$$\mathfrak{P}(w_2) = \rho_4(p_1) + i_2(w_4) \ . \tag{7.3}$$

Note that if $w_2 = 0$ then $\rho_4(p_1)$ is given by $i_2(w_4)$. However, in the resulting Spin case it is more appropriate to use Spin characteristic classes, as we do shortly – see (2.13).
2. Complex bundles. If E is a complex bundle with underlying real bundle $E_{\mathbb{R}}$ and second Chern class $c_2(E)$, then the mod 2 reduction gives $\rho_2(c_2(E)) = w_4(E_{\mathbb{R}})$. The mod 4 reduction is simpler than in the real case, namely

$$\mathfrak{P}(w_4) = \mathfrak{P}(\rho_2(c_2)) = \rho_4(c_2^2) \ . \tag{7.4}$$

3. Real forms. Now consider a bundle E with a real form $E_{\mathbb{R}}$ with corresponding classes $c = c(E)$, $p = p(E_{\mathbb{R}})$, and $w = w(E_{\mathbb{R}})$. Since $p_1(E_{\mathbb{R}}) = 2c_2(E) + c_1(E)^2$, then $\rho_4(c_1^2) = \rho_4(p_1) - \rho_4(2c_2)$.

Relating mod 4 to mod 2 reduction. The mod 4 reduction can be related to the mod 2 reduction through the induced factor map $\kappa_2 : \mathbb{Z}_4 \to \mathbb{Z}_2$ in the exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{i_2} \mathbb{Z}_4 \xrightarrow{\kappa_2} \mathbb{Z}_2 \to 0$ as $\kappa_2 \rho_4 = \rho_2$. Then $\pm 2\rho_4(c_2) = i_2\rho_2(c_2) = i_2(\rho_2(c_2)) = i_2(w_4)$, so that the mod 4 reduction of p_1 satisfies the relation

$$\rho_4(c_1^2) = \rho_4(p_1) + i_2(w_4) \ . \tag{7.5}$$

Special values in dimensions 4 and 8. Now consider an $O(m)$ bundle E over the sphere S^{4k}. Then the Pontrjagin class $p_k(E) \in H^{4k}(S^{4k};\mathbb{Z})$ is divisible by $\epsilon(2k-1)!$, with ϵ being 1 in dimensions $4k = 8m$ and 2 in dimensions $4k = 8m + 4$. So, the first and second Pontrjagin classes of E are divisible by 2 and by 6, respectively. These give rise to generators of BSpin and BString, respectively. Since

$$\rho_4(p_k(E)) = i_{2*}w_{4k}(E) \in H^{4k}(S^{4k};\mathbb{Z}_4) = \mathbb{Z}_4 \ , \tag{7.6}$$

the class $w_{4k}(E)$ is zero if and only if $p_k(E)$ is divisible by 4. This is related to parallelizability and to division algebras [27].

Dimension less than eight. Expression (7.3) is a relation involving the Stiefel-Whitney classes w_2 and w_4. If we find another relation among these two classes then the original expression would then simply relate one Pontrjagin class to one Stiefel-Whitney class. Such simplifications happens in relatively low dimensions. In general, the 4th Wu class is related to the Stiefel-Whitney classes via

$$v_4 = w_4 + w_2^2 \ . \tag{7.7}$$

Note that the Wu class is relevant in relating M-theory to twisted K-theory in type IIA string theory [2], where it is required to admit a lift to twisted cohomology. Now if we are in a situation where $v_4 = 0$ then this gives the desired relation between w_2 and w_4 (assuming neither is zero). This occurs, for example, for any oriented closed manifold of dimension less than eight (i.e. when the second Wu class does not appear for dimension reasons). Of course one could also consider other situations in higher dimensions where this could still happen. Replacing w_4 by w_2^2 in (7.3) gives

$$\begin{aligned} \rho_4(p_1) &= \mathfrak{P}(w_2) + i_2(w_2^2) \\ &= \mathfrak{P}(w_2) + 2\mathfrak{P}(w_2) \ , \end{aligned}$$

so that the new relation for the mod 4 reduction of p_1 is

$$\rho_4(p_1) = -\mathfrak{P}(w_2) \quad \text{when } v_4 = 0. \tag{7.8}$$

Spin^c bundles. For Spin^c bundles, like real oriented bundles, the mod 4 reduction gives

$$\rho_4(p_1(E)) = \mathfrak{P}w_2(E) + i_2(w_4(E)) \ . \tag{7.9}$$

7.2 Fivebrane class mod 8

In this section we will consider the class $\frac{1}{48}p_2$ appearing in the quantization of the dual of the C-field (6.3). The discussion here is analogous to the case of the C-field in section 7.1; instead of reducing mod 2 we should reduce mod 8. However, we found that this case requires extensive discussion and so we will leave it to a separate treatment, and allow ourselves to be content here with the analogy with the String case and with having a novel way of writing the corresponding class.

String characteristic classes. A bundle E with a String structure is characterized by the vanishing of the first Spin characteristic class $Q_1(E) = 0$ (and how it vanishes since it is a homotopy). This has implications on the Stiefel-Whitney classes. Since $\rho_2(Q_1) = w_4$, this implies that $w_4(E) = 0$. Furthermore, since $Sq^3 Q_1 = W_7$, this implies in addition that $W_7(E) = 0$. Analogously to the Spin case, we naturally seek to characterize the corresponding String bundles with characteristic classes of BString. Here also something special happens to the characteristic class defining the structure, namely to the second Pontrjagin class p_2. For String bundles, p_2 is divisible by 6, so that $\frac{1}{3}Q_2$ should be used as a generator instead of simply Q_2 ($= \frac{1}{2}p_2$ when $Q_1 = 0$). Then the first generator for the cohomology of BString will be $\mathcal{Q}_1 := \frac{1}{3}Q_2 = \frac{1}{6}p_2$, and

$$H^*(B\text{String}; \mathbb{Z}) \cong \mathbb{Z}[\mathcal{Q}_1, \mathcal{Q}_2, \cdots] \oplus \mathcal{T} \oplus \mathcal{T}' \; . \tag{7.10}$$

We will not attempt to determine the higher generators $\mathcal{Q}_i \in H^{8i}(B\text{String}; \mathbb{Z})$, $i \geq 2$, nor the 2-torsion $\mathcal{T}$ and the 3-torsion $\mathcal{T}'$, as we are dealing with relatively low dimensions where the only relevant generator is $\mathcal{Q}_1$.

We can characterize the one-loop term $I_8 = \frac{1}{48}(p_2 - \lambda^2)$ [9] [42] (which is part of (6.3)) in the case of a String structure using String characteristic classes $\mathcal{Q}_i$. We did a similar task in [31], where we wrote the one-loop term in the Spin case in terms of the Spin characteristic classes Q_i. We have: *The one-loop term in M-theory on a String manifold is* $\frac{1}{8}\mathcal{Q}_1$. We will study the significance of this elsewhere.

8 Relating the Structures

The structures we have defined in this paper are related. We can find interrelations among them using standard arguments. We have already indicated a few such relations in previous sections, and other relations can be easily deduced. For example, consider a twisted String structure, given by the obstruction $\lambda + \alpha = 0$. Applying the Steendrod square Sq^3 gives $W_7 + Sq^3\alpha = 0$, which is the vanishing of the obstruction to having a String$^{K(\mathbb{Z},3)}$-structure.

Using this type of reasoning, we can relate other structures. We can also talk about further new higher structures (albeit without immediate physical applications). Taking a twisted Fivebrane structure defined via $\frac{1}{6}p_2 + \beta = 0$, and applying the Steenrod square Sq^7 gives

$$Sq^7\left(\frac{1}{6}p_2 + \beta\right) = Sq^7(\mathcal{Q}_1 + \beta) = Sq^7\mathcal{Q}_1 + Sq^7\beta \; , \tag{8.1}$$

which might be considered as an obstruction to defining a *twisted Fivebrane*$^{K(\mathbb{Z},7)}$*-structure*.

This paper has only achieved a first step in uncovering the structures discussed, and there obviously remains a lot of work to study them systematically further. In addition, we hope that geometric and possibly even analytical descriptions of these structures will be possible in the future.

Acknowledgements

The author would like to thank the Max Planck Institute for Mathematics in Bonn for support and for the inspiring atmosphere during the writing of this paper in summer 2010. The author is grateful to Jim Stasheff for useful comments on the draft.

References

[1] L. Auslander and R. Szczarba, *Characteristic classes of compact solvmanifolds*, Ann. of Math. (2) **76** (1962), 1–8.

[2] D. M. Belov and G. W. Moore, *Type II actions from 11-dimensional Chern-Simons theories*, [arXiv:hep-th/0611020].

[3] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958) 458–538.

[4] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, Commun. Math. Phys. **228** (2002) 17-49, [arXiv:hep-th/0106194].

[5] Q. Chen, F. Han, and W. Zhang, *Generalized Witten genus and vanishing theorems*, [arXiv:1003.2325] [math.DG].

[6] E. Diaconescu, D. S. Freed and G. Moore, *The M-theory 3-form and E_8 gauge theory*, Elliptic cohomology, 44–88, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007, [arXiv:hep-th/0312069].

[7] D. Diaconescu, G. Moore, and E. Witten, *E_8 gauge theory, and a derivation of K-theory from M-theory*, Adv. Theor. Math. Phys. **6** (2003) 1031–1134, [arXiv:hep-th/0005090].

[8] C. L. Douglas, *On the twisted K-homology of simple Lie groups*, Topology **45** (2006), 955-988, [arXiv:math/0402082] [math.AT].

[9] M. J. Duff, J. T. Liu, and R. Minasian, *Eleven-dimensional origin of string-string duality: A one loop test*, Nucl. Phys. **B452** (1995) 261-282, [arXiv:hep-th/9506126].

[10] J. Evslin and H. Sati, *Can D-Branes Wrap Nonrepresentable Cycles?*, J. High Energy Phys. **0610** (2006) 050, [arXiv:hep-th/0607045].

[11] M. Flagga and F. Antonsen, *Space-time topology. II. Causality, the fourth Stiefel-Whitney class and space-time as a boundary*, Internat. J. Theoret. Phys. **43** (2004) 1917–1939.

[12] D. S. Freed and E. Witten, *Anomalies in string theory with D-branes*, Asian J. Math. **3** (1999), no. 4, 819–851, [arXiv:hep-th/9907189].

[13] G Hamrick, D Royster, *Flat Riemannian manifolds are boundaries*, Invent. Math. **66** (1982) 405–413.

[14] K. Hori, *Consistency conditions for fivebrane in M-theory on $\mathbb{R}^5/\mathbb{Z}_2$ orbifold*, Nucl. Phys. **B539** (1999) 35-78, [arXiv:hep-th/9805141].

[15] S. Im and H. Kim, *Compact flat manifolds with non-vanishing Stiefel-Whitney classes*, Topology and its Appl. **96** (1999) 267–276.

[16] I. Kriz and H. Sati, *M Theory, type IIA superstrings, and elliptic cohomology*, Adv. Theor. Math. Phys. **8** (2004) 345–395, [arXiv:hep-th/0404013].

[17] I. Kriz and H. Sati, *Type IIB string theory, S-duality and generalized cohomology*, Nucl. Phys. **B715** (2005) 639–664, [arXiv:hep-th/0410293].

[18] I. Kriz and H. Sati, *Type II string theory and modularity*, J. High Energy Phys. **08** (2005) 038, [arXiv:hep-th/0501060].

[19] J. Kalkkinen, *Global spinors and orientable five-branes*, J. High Energy Phys. **0609** (2006) 028, [arXiv:hep-th/0604081].

[20] T. P. Killingback, *Global anomalies, string theory and spacetime topology*, Class. Quantum Grav. **5** (1988), no. 9, 1169–1185.

[21] H.B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, NJ, 1989.

[22] W. Lerche, B. E. W. Nilsson, A. N. Schellekens, and N. P. Warner, *Anomaly cancelling terms from the elliptic genus*, Nucl. Phys. **B 299** (1988), no. 1, 91–116.

[23] V. Mathai, M. K. Murray, and D. Stevenson, *Type I D-branes in an H-flux and twisted KO-theory*, J. High Energy Phys. **0311** (2003) 053, [`arXiv:hep-th/0310164`].

[24] C. R. F. Maunder, *Algebraic Topology*, Dover Publications, Inc., Mineola, New York, 1996.

[25] J. P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press, 1999.

[26] D. A. McLaughlin, *Local formulae for Stiefel-Whitney classes*, Manuscripta Math. **89** (1996) 1–13.

[27] J. Milnor, *Some consequences of a theorem of Bott*, Ann. of Math. (2) **68** (1958) 444–449.

[28] R. E. Mosher and M. C. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, Harper & Row, Publishers, New York-London 1968.

[29] D. Quillen, *The mod 2 cohomology rings of extra-special* 2*-groups and the spinor groups*, Math. Ann. **194** (1971) 197–212.

[30] H. Sati, *A higher twist in string theory*, J. Geom. Phys. **59** (2009) 369-373, [`arXiv:hep-th/0701232`].

[31] H. Sati, *An Approach to anomalies in M-theory via KSpin*, J. Geom. Phys. **58** (2008) 387-401, [`arXiv:0705.3484`] [`hep-th`].

[32] H. Sati, *Geometric and topological structures related to M-branes*, to appear in proc. Symp. Pure Math., [`arXiv:1001.5020`] [`math.DG`].

[33] H. Sati, *Geometry of Spin and Spin^c structures in the M-theory partition function*, [`arXiv:1005.1700`] [`hep-th`].

[34] H. Sati, *Geometric and topological structures related to M-branes II: Twisted String and String^c structures*, [`arXiv:1007.5429`] [`hep-th`].

[35] H. Sati, U. Schreiber and J. Stasheff, *L_∞-connections and applications to String- and Chern-Simons n-transport*, in *Recent Developments in QFT*, eds. B. Fauser et al., Birkhäuser, Basel (2008), [`arXiv:0801.3480`] [`math.DG`].

[36] H. Sati, U. Schreiber, and J. Stasheff, *Fivebrane structures*, Rev. Math. Phys. **21** (2009), no. 10, 1197–1240, [`arXiv:math/0805.0564`] [`math.AT`].

[37] H. Sati, U. Schreiber, and J. Stasheff, *Twisted differential String- and Fivebrane structures*, [`arXiv:0910.4001`] [`math.AT`].

[38] R. E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, NJ 1968.

[39] A. Strominger, *Open p-branes*, Phys. Lett. **B 383** (1996) 44–47, [`arXiv:hep-th/9512059`].

[40] E. Thomas, *On the cohomology groups of the classifying space for the stable spinor groups*, Bol. Soc. Mat. Mexicana (2) **7** (1962) 57–69.

[41] P. K. Townsend, *D-branes from M-branes*, Phys. Lett. **B373** (1996) 68–75, [`arXiv:hep-th/9512062`].

[42] C. Vafa and E. Witten, *A One-loop test of string duality*, Nucl. Phys. **B447** (1995) 261–270, [`arXiv:hep-th/9505053`].

[43] A. T. Vasquez, *Flat Riemannian manifolds*, J. Differential Geom. **4** (1970) 367–382.

[44] B.-L. Wang, *Geometric cycles, index theory and twisted K-homology*, J . Noncommut. Geom. **2** (2008), 497–552, [`arXiv:0710.1625`] [`math.KT`].

[45] E. Witten, *Global anomalies in string theory*, Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), 61–99, World Sci. Publishing, Singapore, 1985.

[46] E. Witten, *On flux quantization in M-theory and the effective action*, J. Geom. Phys. **22** (1997) 1-13, [`arXiv:hep-th/9609122`].

[47] E. Witten, *D-Branes and K-theory*, J. High Energy Phys. **12** (1998) 019, [`arXiv:hep-th/9810188`].

[48] E. Witten, *Duality relations among topological effects in string theory*, J. High Energy Phys. **0005** (2000) 031, [`arXiv:hep-th/9912086`].